A logical investigation of heterogeneous reasoning with graphs in elementary economics (Extended version)

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Abstract

Heterogeneous reasoning, which combines various sentential, diagrammatic, and graphical representations, is a salient component of logic, mathematics, and computer science. Another remarkable field in which it is applied is economics. In economics, various functions such as supply and demand functions are represented by graphs, which are used to explain and derive economic laws. In this paper, we apply proof-theoretic techniques developed in previous studies to heterogeneous reasoning with graphs in elementary economics. We apply the natural deduction-style formalization to heterogeneous reasoning with graphs. We also present a proof-theoretic analysis of free rides, and analyze the efficiency of heterogeneous reasoning with graphs. We further discuss abductive reasoning in elementary economics in the context of heterogeneous reasoning.

Contents

1 Introduction

In general, reasoning is carried out by combining various sentential, diagrammatic, and graphical representations according to the given situation. Such reasoning is called heterogeneous reasoning, and its importance has been known since the earliest research on diagrammatic reasoning. Barwise and Etchemendy introduced a heterogeneous system combining graphical representations and first-order formulas in a block world [3, 1]. Heterogeneous systems based on Venn, Euler, and spider diagrams and first-order formulas have also been formalized, and their implementations have been developed, e.g., [6, 17, 23]. We studied heterogeneous reasoning with correspondence tables to solve certain scheduling problems in [20, 21]. Furthermore, the computational architecture of heterogeneous reasoning was investigated in [2], and a general framework for a heterogeneous reasoning theorem prover was studied in [24]. See [1] for early examples of heterogeneous systems.

Heterogeneous reasoning is a salient component of logic, mathematics, and computer science. Another remarkable field it applies to is economics. When one opens an economics textbook, one finds a number of graphs representing various mathematical functions in economics such as supply and demand functions. This may be because graphs make economic laws and theory visually accessible to novices and students who are not comfortable with mathematics.

In this paper, we apply the proof-theoretic techniques developed in our previous studies [20, 21, 19] to heterogeneous reasoning with graphs in elementary economics. We apply the natural deduction-style formalization, which makes it possible to apply well-developed proof-theoretic techniques to the analysis of heterogeneous reasoning with graphs. We also apply the proof-theoretic analysis of free rides developed in [19], and analyze the efficiency of heterogeneous reasoning with graphs. We further discuss abductive reasoning in elementary economics. Abduction is not restricted to economic reasoning, but appears throughout scientific reasoning. It was first formalized by the philosopher Charles Sanders Peirce in the 19th century, and is distinct from other types of reasoning, i.e., deduction and induction. Peirce defined abduction as the inference process of finding a hypothesis that explains a given observation, and it is often described as "inference to the best explanation." Abduction has been discussed by philosophers and logicians, and has been extensively studied in the literature on artificial intelligence (see, for example, [13, 4] for surveys of abduction in artificial intelligence). In the context of heterogeneous reasoning, we are able to formalize abductive reasoning in elementary economics in the style we employ in our actual reasoning.

Economic reasoning similar to that discussed here has been investigated in the framework of qualitative reasoning, e.g., [5, 9]. The study of qualitative reasoning originally evolved from the investigation of physical reasoning, where qualitative information is exploited without exact quantitative information or precise values (see [7]). Basic economic concepts such as the law of demand, which states that 'a higher price leads to a smaller quantity demanded,' are usually expressed in abstract qualitative forms without specifying any precise mathematical function. Thus, economic reasoning has been investigated as an application of qualitative reasoning. Additionally, diagrammatic reasoning does not generally need exact values or quantitative information, and is therefore also compatible with qualitative reasoning. Thus, the relationship between the two has been examined [22, 18, 8], and qualitative spatial reasoning has recently been investigated within the framework of the *region connection calculus (RCC)* [12, 14]. Furthermore, from a cognitive science viewpoint, the CaMeRa (computational model of multiple representations) approach to heterogeneous reasoning with graphs in elementary economics has been proposed by [18]. In this article, each step of economic reasoning with graphs, as formalized by qualitative reasoning studies, is associated with the cognitive steps in their model. We extend their framework and investigate heterogeneous reasoning in economics from the viewpoint of logic.

In Section 2, we introduce an example of economic reasoning with graphs, and then, we investigate the structure of such reasoning. In Section 3, we formalize our heterogeneous logic with graphs in elementary economics HLGe, and we demonstrate its soundness. We further investigate some properties of HLGe, and analyze free rides in the system. Finally, in Section

4, we discuss abductive reasoning in elementary economics is formalized by slightly modifying our HLGe.

2 Reasoning with graphs in economics

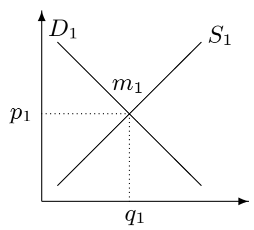
2.1 Example

Let us examine the following example of reasoning with graphs in elementary economics, which is a slight modification of an example given in [10] written by Krugman and Wells.

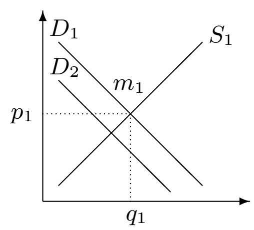
Example 2.1 ([10] p.94) When a new, faster computer chip is introduced, (1) demand for computers using the older, slower chips decreases. (This graphically corresponds to a leftward shift of the demand curve from the original D_1 to D_2, which we express as $D_2 \leftarrow D_1$.) Simultaneously, (2) computer makers increase their production of computers containing the old chips in order to clear out their stocks of old chips. (Graphically, this corresponds to a rightward shift of the supply curve from the original S_1 to S_2; $S_1 \rightarrow S_2$.) Furthermore, (3) it is widely known that there is only a minor change in the new computer chip, and it does not make computers dramatically faster. That is, the decrease in demand is small relative to the increase in supply. What happens to the equilibrium price and quantity of computers?

Relationships between quantity demanded and price, as well as between quantity supplied and price, are represented mathematically by functions of price, i.e., a demand function and a supply function, respectively. In economics, graphs of these demand and supply functions are conventionally drawn in a two-dimensional plane, where the vertical axis represents price and the horizontal axis represents quantity demanded or supplied. (Note that this is not consistent with mathematical convention, where the horizontal axis normally represents the independent variable, i.e., price.) The law of demand says that a higher price leads to a smaller quantity demanded; hence, demand curves generally slope downward. Similarly, the law of supply says that a higher price leads to a larger quantity supplied, and so supply curves generally slope upward.

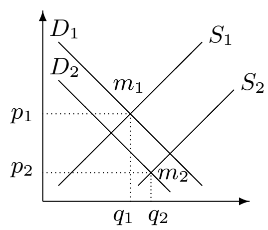
In the above example, there are no concrete demand and supply functions. Note that it is impossible to draw an "arbitrary" curve, since a drawn curve is more or less specific, and it may show something not derived in general. (This property of diagrams is called *over-specificity* by Shimojima [15].) Hence, we draw demand and supply curves in the simplest manner, i.e., as straight lines with slopes of -1 and 1, respectively, as in the following G_1. We assume that G_1 represents the initial state of the given market, and its equilibrium is $m_1(q_1, p_1)$.



G_1



G_2



G_3

From premise (1), the quantity demanded in this market decreases, and the original demand curve D_1 shifts to D_2, as in G_2. From premise (2), the quantity supplied increases, and the supply curve S_1 shifts to S_2, as in G_3. Although we do not know how much D_1 (resp. S_1) shifts to D_2 (resp. S_2), we can infer from premise (3) that the horizontal shift of the supply curve is greater than that between D_1 and D_2, as expressed in G_3. These shifts in the demand and supply curves lead to the new equilibrium $m_2(q_2, p_2)$. By comparing m_2 and the original m_1, we find $q_1 < q_2$ and $p_1 > p_2$, i.e., the equilibrium quantity rises and the equilibrium price falls. Note that we obtain this result without loss of generality, even if we do not know the exact shifts in the demand and supply curves, if the ordering condition (3) holds, and if typical curves of slopes ± 1 are assumed.

Remark 2.2 Without condition (3) on the horizontal width of the shifts, the relation between q_1 and q_2 is not uniquely determined by the shift widths of the supply and demand curves, although $p_1 > p_2$ generally holds. (Cf. Example 3.11.)

Remark 2.3 In general, what is derived about p and q depends on shift sizes and slopes of demand and supply curves. However, it is shown that if we restrict demand and supply curves to be straight lines whose absolute values of slopes are equal, then what is derived depends only on shift sizes of the curves. Thus, strictly speaking, we solved the above question in Example 2.1 graphically, under the following additional assumption:
(4) Demand/supply decreases/increases linearly in the same rate.

2.2 Structure of reasoning with graphs in economics

Based on the previous example, let us investigate the structure of reasoning with graphs in elementary economics. The reasoning in Example 2.1 goes as follows.

1. An appropriate graph is given, which describes the initial state of a market.
2. We shift a curve based on the given premise, which represents an increase or decrease in demand or supply. This step may be repeated several times. This shifting operation may be considered as the addition of a new curve, since it is convenient to keep the original curve to compare equilibriums at a later point.
3. With this shift in a curve, a new intersection (equilibrium) arises between the demand and supply curves.
4. We compare the new intersection and the original one, and read off the changes in price and quantity.

Let us compare the above graphical reasoning with algebraic reasoning, where we solve simultaneous equations describing given demand and supply functions.

1. Let the given demand function D_1 be $y = -x + \gamma$, and the supply function S_1 be $y = x + \delta$, where γ, δ are real numbers.

   Note that, in accordance with our graphs in elementary economics, we assume slopes of D_1 and S_1 be -1 and 1, respectively.

   (In fact, the above D_1 (and S_1) is the inverse of the demand function from the viewpoint of economics, since y represents price and x represents quantity demanded. However, in our simple and typical setting, there is technically no difference between a demand function and its inverse, i.e., we can freely replace x and y.)

2. For some real numbers $\alpha > 0$ and $\beta > 0$ such that $\alpha < \beta$, D_2 can be expressed as $y = -x + \gamma - \alpha$, and S_2 as $y = x + \delta - \beta$.
3. By solving the simultaneous equations D_1 and S_1, we find $q_1 = \frac{\gamma-\delta}{2}$ and $p_1 = \frac{\gamma+\delta}{2}$, which represent the original equilibrium quantity and price.
4. Similarly, by solving D_2 and S_2, we find $q_2 = \frac{\gamma-\delta-\alpha+\beta}{2}$ and $p_2 = \frac{\gamma+\delta-\alpha-\beta}{2}$, which represent the new equilibrium quantity and price.
5. By comparing the equilibrium quantities, we find that $q_1 - q_2 = \frac{\alpha-\beta}{2} < 0$ (since $\alpha < \beta$), and hence, we have $q_1 < q_2$.
6. By comparing the equilibrium prices, we find $p_1 - p_2 = \frac{\alpha+\beta}{2} > 0$, i.e., $p_1 > p_2$.

Although the above calculation is not difficult, it is slightly cumbersome compared with our graphical reasoning, where no mathematical knowledge is required. Furthermore, if we formalize it in the framework of mathematical logic, a considerable number of steps are required. (See, for example, [11] for a formalization of arithmetic.)

Economic reasoning similar to our example has been studied in the framework of qualitative reasoning, e.g., [5, 9]. Qualitative reasoning studies investigate reasoning based on qualitative information, instead of precise quantitative information [7]. In qualitative reasoning studies, with the aim of implementation, economic reasoning "without graphs" is investigated. In [5] (among others), economic laws such as "a higher price leads to a smaller (resp. larger) quantity demanded (resp. supplied)" and "an increase in demand raises an excess demand" are formalized as *causal relations*. Based on such statements, economic reasoning is then formalized exactly as one traces the points of intersection between demand and supply curves in a graph. Thus, the reasoning formalized in qualitative reasoning studies is considered as another symbolic or linguistic counterpart of our graphical reasoning.

In some aspects, the economic reasoning we investigate here is an extension of previous research, where either the demand or supply curve is allowed to shift just once. Such an analysis has been extended to a more complicated, multivariable setting in [5]. However, we concentrate on analyzing the basic demand and supply market, but allow *simultaneous shifts* of the demand and supply curves as in Example 2.1.

3 Heterogeneous logic with graphs in economics HLGe

3.1 Syntax and semantics

In this section, we introduce heterogeneous logic with graphs in elementary economics, HLGe. In this paper, we assume that demand and supply curves are straight lines, and their slopes are -1 and 1, respectively. Then, as seen in the previous example, exactly what can be derived depends on the size of the shift in a demand or supply curve. Hence, we assume the shift size is specified when we consider the shift in a curve. However, in our qualitative framework, the exact value of the shift is not as significant as the relation between the magnitude of the shifts. Thus, we do not express the shift size as an exact numeral, but as a constant a that represents some real number. A formula such as $C \xrightarrow{a} C'$ then means "the curve C shifts rightward to C' with shift width a."

For our heterogeneous system HLGe, we use the following symbols with subscripts when necessary:

- Logical connectives: $\&, \vee, \Rightarrow, \Leftrightarrow, \neg, \forall, \exists$
- Constants for widths: a, b, c
- Constants for coordinates: p, q, r
- Variables for coordinates: x, y
- Curves: D, S, C, B

We also use typical mathematical function symbols such as $+$ and $-$, and predicate symbols such as $=$ and $<$.

Among the usual mathematical formulas, we distinguish the following special formulas in HLGe. A **demand (resp. Supply) curve** is written as $D(x) = -x + r$ (resp. $S(x) = x + r$) for some r. When (q, p) is an **intersection point** or **equilibrium** of C and C', we write $C \cap C'(q) = p$. We define **shift formulas** as follows:

- $D \stackrel{a}{\to} D' := \forall x(D(x) = -x + r \Leftrightarrow D'(x) = -x + r + a)$
- $D' \stackrel{a}{\leftarrow} D := \forall x(D(x) = -x + r \Leftrightarrow D'(x) = -x + r - a)$

  Similarly for $S \stackrel{a}{\to} S'$ and $S' \stackrel{a}{\leftarrow} S$.

We denote formulas by φ, ψ.

We define graphs in elementary economic reasoning as follows.

Definition 3.1 (Graph) A **graph** in HLGe consists of the following items:

- The first quadrant of the xy-coordinate space.
- Straight lines of slope 1, called **supply curves** and named $S, S', S_1, \ldots$; and of slope -1, called **demand curves** and named $D, D', D_1, \ldots$.

  When we do not distinguish between supply and demand, we denote a curve by $C, C', C_1, \ldots$.
- Every point of intersection of straight lines is accompanied by its coordinates.

Although we consider only straight lines, we call them *curves* following the convention of economics. In principle, every point of intersection is accompanied by its coordinates, but we sometimes omit them to avoid visual complexity.

We define the *width* between two lines in a given graph.

Definition 3.2 (Width) Let C_i and C_j be a pair of lines that are parallel in a graph. Let q_i (and q_j) be the intersection point of C_i (resp. C_j) and the vertical axis when C_i (resp. C_j) is extended as necessary. We define the **width** $w(C_i, C_j)$ between C_i and C_j as $|q_i - q_j|$. When G is a graph, by $w(G)$, we denote the set of all widths in G.

Obviously, we have $w(C, C') = a$ for $C \stackrel{a}{\to} C'$. Note that the above *width* is not the geometrical distance between two lines, but it is the difference between the y-intercepts of the given lines. In this paper, we do not consider the arithmetic of widths; hence, we do not consider widths such as $a \times b$ and $\frac{1}{2}a$ for widths a, b.

In contrast to a graph drawn as a diagram, we consider the *type* of a graph, which is a symbolic specification. The type of a graph also defines what kind of information we can extract from it; cf. our inference rule Observe in Definition 3.10. As in graph theory, we usually do not distinguish between a drawn graph and its type, and denote both of them by G.

Definition 3.3 (Type) The **type of a graph** G is $(\mathcal{C}, l_w, \mathcal{E}, l_p, l_q)$, where:

- $\mathcal{C}$ is two sequences $D_i \to D_j \to \cdots \to D_n$; $S_k \to S_l \to \cdots \to S_m$ of demand curves and supply curves in G, respectively, which are ordered from left to right as they are in the drawn graph G.
- By allowing equality, i.e., some elements are equal, l_w is the linearly ordered set $w(C_i, C_j) < \cdots < w(C_k, C_l) < \cdots$ of all widths in G.
- $\mathcal{E}$ is the set of points of intersection in G of the form $D_i \cap S_j(q_k) = p_k$.
- l_p is the linearly ordered set $p_i < p_j < \cdots$ of all y-coordinates of intersections.
- l_q is the linearly ordered set $q_i < q_j < \cdots$ of all x-coordinates of intersections.

Actually, the sequence $\mathcal{C}$, and the linear ordering of widths l_w, and intersections $\mathcal{E}$, which define their coordinates as well, are sufficient to specify a graph. (Cf. Definition 3.8 for our interpretation of graphs.)

Example 3.4 The type of G in the following Example 3.6 is as follows:

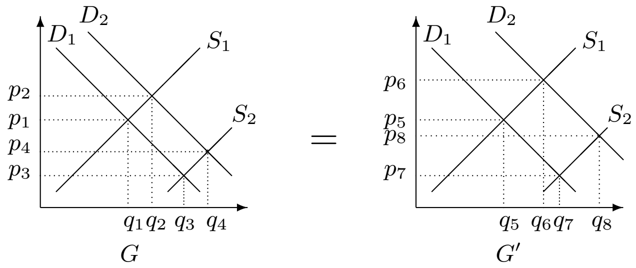
- $\mathcal{C} = D_1 \to D_2 \ ; \ S_1 \to S_2$
- $l_w = w(D_1, D_2) < w(S_1, S_2)$
- $\mathcal{E} = \{D_1 \cap S_1(q_1) = p_1, D_2 \cap S_1(q_2) = p_2, D_1 \cap S_2(q_3) = p_3, D_2 \cap S_2(q_4) = p_4\}$
- $l_p = p_3 < p_4 < p_1 < p_2$
- $l_q = q_1 < q_2 < q_3 < q_4$

Based on types of graphs, we define the equivalence between graphs. Note that points of intersection are determined when curves are given. Furthermore, in our qualitative framework, the exact width values, as well as the xy-coordinates of the intersections, are not significant, whereas the ordering relation among them is important. Thus, we regard two graphs as being equivalent when their arrangement of curves and their ordering relation among widths are equivalent.

Definition 3.5 Two graphs $G = (\mathcal{C}, l_w, \mathcal{E}, l_p, l_q)$ and $G' = (\mathcal{C}', l'_w, \mathcal{E}', l'_p, l'_q)$ are **equivalent**, written as $G = G'$, if $\mathcal{C} = \mathcal{C}'$ and $l_w = l'_w$.

Note that if $\mathcal{C} = \mathcal{C}'$, then l_w and l'_w consist of the same widths $w(C_i, C_j)$, although the values of $w(C_i, C_j)$ of l_w and of l'_w may be different. Thus, $l_w = l'_w$ means that the ordering relation, not the exact value, is the same among the widths of l_w and l'_w.

Example 3.6 (Equivalence) The following graphs G and G' are equivalent, since we have $\mathcal{C} = \mathcal{C}' = D_1 \to D_2; S_1 \to S_2$, and $l_w = l'_w = w(D_1, D_2) < w(S_1, S_2)$ even though the value of $w(D_1, D_2)$ in G is smaller than that in G'.



The translation of our graphs into first-order formulas is straightforward based on the type of graph.

Definition 3.7 (Translation of graphs) A graph G of $(\mathcal{C}, l_w, \mathcal{E}, l_p, l_q)$ is translated into a conjunctive formula $\bigwedge \mathcal{C} \,\&\, \bigwedge l_2 \,\&\, \bigwedge \mathcal{E} \,\&\, \bigwedge l_p \,\&\, \bigwedge l_q$, where $\bigwedge X$ denotes the conjunction of all corresponding formulas contained in the set X.

For the set-theoretical semantics of HLGe, it is sufficient to employ a domain of real numbers in which arithmetic operations such as $+$ and $-$ are defined. Hence, we provide the real closed field $\mathbb{R}$ with the ordering relation $<$ as our model. In the following, we implicitly assume an interpretation, and do not distinguish between a symbol, say C, and its interpretation, f^C. Formulas of HLGe are interpreted as usual. For example, a formula expressing an intersection $C \cap C'(q_1) = p_1$ is interpreted in a model M as follows.

$$M \models C \cap C'(q_1) = p_1 \text{ if and only if } (q_1, p_1) \in C \cap C'.$$

Graphs in HLGe are interpreted as follows.

Definition 3.8 (Interpretation of graphs) Let M be a model. Let G be a graph of $(\mathcal{C}, l_w, \mathcal{E}, l_p, l_q)$, where $\mathcal{C} = D_1 \to D_2 \to \cdots D_n;\ S_1 \to S_2 \to \cdots S_m$, and $l_w = w(C_1, C_2) < w(C_3, C_4) < \cdots < w(C_k, C_l)$. Then, $M \models G$ if and only if

- $M \models D_1 \xrightarrow{w(D_1,D_2)} D_2 \,\&\, \cdots \,\&\, D_{n-1} \xrightarrow{w(D_{n-1},D_n)} D_n$; and
- $M \models S_1 \xrightarrow{w(S_1,S_2)} S_2 \,\&\, \cdots \,\&\, S_{m-1} \xrightarrow{w(S_{m-1},S_m)} S_m$; and
- $M \models w(C_1, C_2) < w(C_3, C_4) < \cdots < w(C_k, C_l)$; and
- $M \models \mathcal{E}$, that is, $M \models C \cap C'(q) = p$ for all $C \cap C'(q) = p \in \mathcal{E}$.

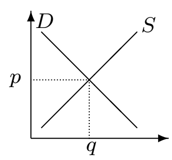
Example 3.9 For example, we can consider the following semantic consequence, which corresponds to Example 2.1:

$$D \xrightarrow{a} D' \ ,\ S' \xleftarrow{b} S \ ,\ a < b \ ,\ D \cap S(q_1) = p_1 \ ,\ D' \cap S'(q_2) = p_2 \ \models\ p_1 > p_2 \ \&\ q_1 < q_2$$

3.2 Inference rules of HLGe

We introduce inference rules for HLGe that are defined as an extension of natural deduction for the usual mathematics. We assume the usual axioms of the real closed field, and of the ordering $<$. Furthermore, we assume the following axioms of our elementary economic reasoning.

- We consider only the first quadrant of the xy-coordinate space, that is, every curve is restricted to the first quadrant.
- We assume that there always exists an intersection point in the first quadrant of the xy-coordinate space, for any pair of curves that are not parallel.
- We assume the following form of graphs as axioms, which is a graphical representation of the formula: there exists two linear functions $D(x) = -x + p_1$ and $S(x) = x + p_2$ for some p_1 and p_2 such that $D \cap S(q) = p$ with $q, p \geq 0$.



The inference rules for HLGe consist of the usual natural deduction rules for first-order formulas and rules for graphs. Cf. [20]. We now define our inference rules for graphs in HLGe. According to Plummer-Etchemendy [2], inference rules characteristic of heterogeneous systems are generally called *transfer rules*, and allow the transfer of information from one form of representation to another. Typical rules in Hyperproof [3, 1] are Apply (from sentences to a diagram) and Observe (from a diagram to sentences). Our Apply has the following form:

$$\frac{G \quad C \xrightarrow{a} C' \quad l}{G'} \qquad \text{or} \qquad \frac{G \quad C' \xleftarrow{a} C \quad l}{G'}$$

where G' is obtained from a given graph G by adding a new curve, say C', and l specifies a linear ordering of all widths (including a) in G'. We read this rule as: "we apply $C \xrightarrow{a} C'$ to amplify G to G'," or "we extend G to G' by adding the new information of $C \xrightarrow{a} C'$ (or $C' \xleftarrow{a} C$) to G."

Our Apply can only be used when all of the ordering relations between the shift width of an additional curve and the widths already in G are specified by the above l. Thus, when the relations are not fully specified in the given premise, we need to enumerate all possible cases. (See the following rule of Cases.)

Definition 3.10 Inference rules for graphs of HLGe consist of the following Apply and Observe.

Apply: Let G be a graph, that contains a curve C but does not contain C'. Let $C \xrightarrow{a} C'$ be a shift formula. Let l be an ordering condition that specifies a linear ordering of all widths in $w(G) + C' = w(G) \cup \{w(C', B) \mid B \text{ is a curve parallel to } C \text{ (including } C) \text{ in } G\}$:

$$\frac{G \quad C \xrightarrow{a} C' \quad l}{G'} \ \mathsf{Apply}$$

where G' is obtained from G by adding the curve C' so that (1) C' is parallel to C; (2) C' is orthogonal to every curve that is orthogonal to C; (3) the width between C' and C is a; (4) the widths including a satisfy l.

Similarly for $C' \xleftarrow{a} C$.

Observe: From a given graph G, we can extract, as a conclusion, any corresponding formula contained in the type of G.

When the given ordering condition l does not fully specify a linear ordering among $w(G) + C'$, we cannot apply Apply. In such a case, we enumerate all possible linear orderings of $w(G) + C'$ and apply the $\vee$-elimination rule ($\vee E$) of natural deduction: Let $\{l_1, \ldots, l_n\}$ be the enumeration of all possible linear orderings of $w(G) + C'$ that satisfies the given l. (Note that the widths $w(G)$ is already linearly ordered, since all shift widths are fixed in a given graph G. Thus, we can obtain $l_1, \ldots, l_n$ by applying the usual insertion sort algorithm.) Since $l_1 \vee \cdots \vee l_n$ is provable from l, we divide the case according to $l_1, \ldots, l_n$ by using $\vee E$, and then, apply Apply in every case as follows:

Cases:

$$\dfrac{\dfrac{l}{l_1 \vee \cdots \vee l_n} \qquad \begin{array}{c} \dfrac{G \quad C \xrightarrow{a} C' \quad [l_1]^m}{G_1}\ \mathsf{Apply} \\ \vdots \\ G'/\psi \end{array} \qquad \cdots \qquad \begin{array}{c} \dfrac{G \quad C \xrightarrow{a} C' \quad [l_n]^m}{G_n}\ \mathsf{Apply} \\ \vdots \\ G'/\psi \end{array}}{G'/\psi} \ \vee E, m$$

where G'/ψ denotes that either a graph G' or a first-order formula ψ is obtained, and $[l_i]^m$ denotes the assumption l_i is closed as usual in natural deduction. A double line is an abbreviation of some applications of rules. By regarding the above part of a proof as an inference rule, we call it the rule of Cases.

In an application of Apply, we assume that there always exists an intersection between the additional curve and every orthogonal curve already appearing in a given premise graph. This is feasible, as we can always arrange the positions of curves by retaining ordering relations among widths under the equivalence of graphs in our qualitative framework.

The notion of proof in HLGe is defined inductively, as in natural deduction. We use the symbol $\vdash$ to denote the provability relation in HLGe.

Example 3.11 (A proof in HLGe) Fig. 1 is an example of a proof in HLGe of the following consequence, which describes the situation of Example 2.1 without the condition (3) describing the shift widths of supply and demand curves.

$$D_2 \overset{a}{\leftarrow} D_1 \ , \ S_1 \overset{b}{\rightarrow} S_2 \ , \ D_1 \cap S_1(p_1) = q_1 \ , \ D_2 \cap S_2(p_2) = q_2 \ \vdash \ p_1 > p_2$$

In addition to the above example, the following are also provable in HLGe for example:

- $D_1 \overset{a}{\rightarrow} D_2 \ \vdash \ p_1 < p_2 \,\&\, q_1 < q_2$;
- $D_1 \overset{a}{\rightarrow} D_2 \ , \ S_1 \overset{b}{\rightarrow} S_2 \ \vdash \ q_1 < q_2$;
- $D_1 \overset{a}{\rightarrow} D_2 \ , \ S_1 \overset{b}{\rightarrow} S_2 \ , \ a < b \ \vdash \ p_1 > p_2 \,\&\, q_1 < q_2$,

  and so on.

We now show that HLGe can handle simultaneous curve shifts even though Apply and Cases are applied in order during a proof.

Proposition 3.12 (Simultaneous) *Consider a graph G in which D_1, S_1 appear but D_2, S_2 do not. Then, the same set of graphs is obtained from G in the following two cases:*

1. *We first apply* Apply *(or* Cases*) to $D_1 \overset{a}{\rightarrow} D_2$, and then to $S_1 \overset{b}{\rightarrow} S_2$;*
2. *We first apply* Apply *(or* Cases*) to $S_1 \overset{b}{\rightarrow} S_2$, and then to $D_1 \overset{a}{\rightarrow} D_2$.*

Proof. We denote by $(G + D_2) + S_2$ the set of graphs obtained by the above case (1), and by $(G + S_2) + D_2$ those obtained by (2). In either case, we obtain graphs consisting of the same curves. Thus, it is sufficient to show that orderings of widths of $(G + D_2) + S_2$ and of $(G + S_2) + D_2$ are equivalent. Let $L(w(G) + C)$ be the enumeration of all possible linear orderings of $w(G) + C$ that satisfies the given ordering conditions in premises. Then, we show $L(w(G + D_2) + S_2) = L(w(G + S_2) + D_2)$, which is obtained by returning to the definition.

Since $w(G) + D_2 = w(G) \cup \{w(D_2, B) \mid B \in G\}$, we have $w(G + D_2) + S_2 = w(G) \cup \{w(D_2, A) \mid A \in G\} \cup \{w(S_2, B) \mid B \in G + D_2\} = w(G) \cup \{w(D_2, A) \mid A \in G\} \cup \{w(S_2, B) \mid B \in G\} \cup \{w(S_2, D_2)\}$.

Similarly, since $w(G) + S_2 = w(G) \cup \{w(S_2, B) \mid B \in G\}$, we have $w(G + S_2) + D_2 = w(G) \cup \{w(S_2, B) \mid B \in G\} \cup \{w(D_2, A) \mid A \in G + S_2\} = w(G) \cup \{w(S_2, B) \mid B \in G\} \cup \{w(D_2, A) \mid A \in G\} \cup \{w(D_2, S_2)\}$.

Therefore, we have $L(w(G + D_2) + S_2) = L(w(G + S_2) + D_2)$. ■

We now establish the soundness theorem for HLGe. After dividing several cases, this is proved in a similar way to that described in Section 2.2 by algebraic calculation.

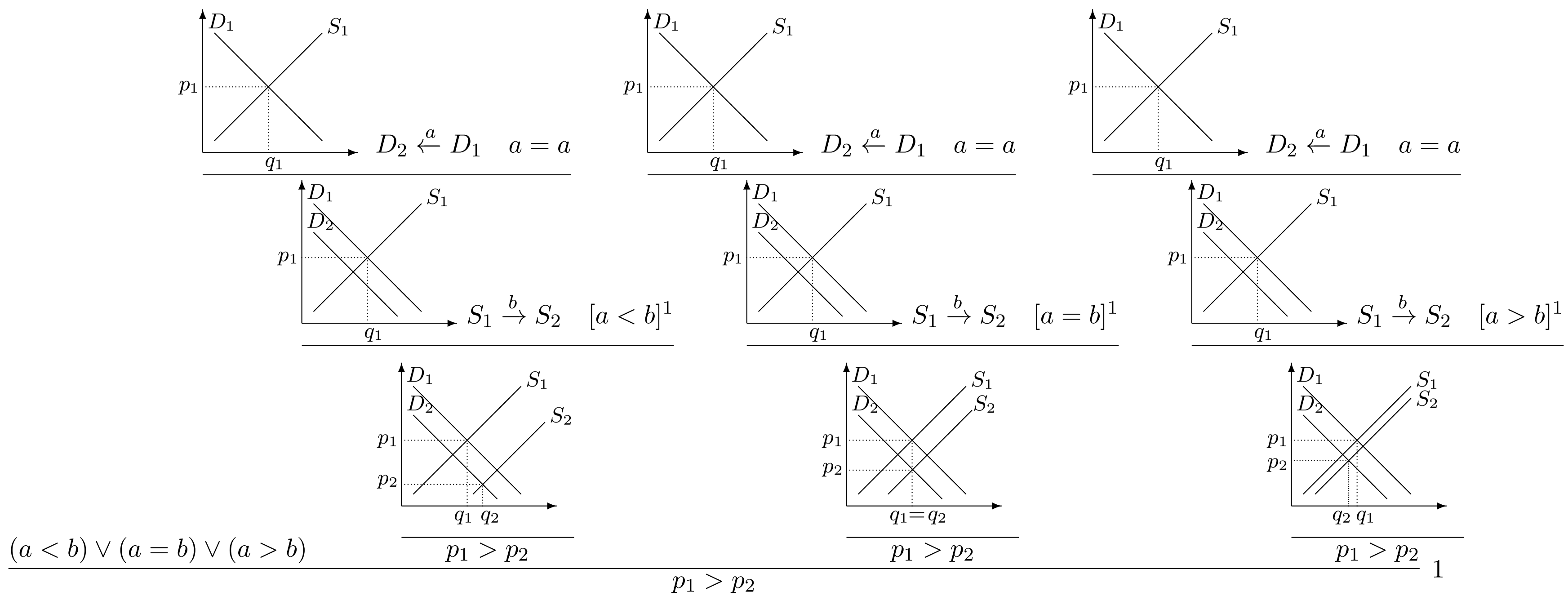


Fig. 1 A proof of $D_2 \overset{a}{\leftarrow} D_1$, $S_1 \overset{b}{\rightarrow} S_2$, $D_1 \cap S_1(p_1) = q_1$, $D_2 \cap S_2(p_2) = q_2 \vdash p_1 > p_2$.
Note that $(a < b) \vee (a = b) \vee (a > b)$ and $a = a$ are axioms of HLGe.
Observe that the ordering relation between q_1 and q_2 is not uniquely determined.

Theorem 3.13 (Soundness) *Let $\mathcal{S}$ be a set of shift formulas; $\mathcal{E}$ be a set of intersections; $\mathcal{O}$ be a set of ordering conditions among widths; and $\mathcal{A}$ be a conjunction of formulas comparing x- and y-coordinates. If $\mathcal{S}, \mathcal{E}, \mathcal{O} \vdash \mathcal{A}$ (provable), then $\mathcal{S}, \mathcal{E}, \mathcal{O} \models \mathcal{A}$ (semantically valid).*

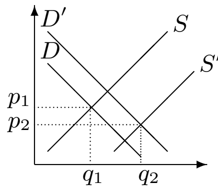
Proof. The theorem is shown by induction on the length of proofs in the usual manner. We show the following case for Apply (the other cases are similar):

$$\frac{G \quad D \xrightarrow{a} D' \quad l}{G'}\ \mathsf{Apply}$$

where D appears in G and D' does not, and $l \in L(w(G)+D')$. Assume $M \models G \& (D \xrightarrow{a} D') \& l$. We show $M \models G'$. Let the type of G be $(\mathcal{C}, l_w, \mathcal{E}, l_p, l_q)$, and that of G' be $(\mathcal{C}', l'_w, \mathcal{E}', l'_p, l'_q)$. Since $M \models G \,\&\, (D \xrightarrow{a} D') \,\&\, l$, we have $M \models \mathcal{C}'$ and $M \models l'_w$. We now show that $M \models \mathcal{E}'$, that is, we can define every new intersection (p_i, q_i) in M after the addition of the curve D', and $M \models l'_p$ and $M \models l'_q$, that is, if $p_i < p_j$ holds in G', then we also have $p_i < p_j$ in M semantically.

Let $D \cap S(q_1) = p_1$ already exist in G, and $D' \cap S'(q_2) = p_2$ be a new intersection point for some S' in G. Let D be $y = -x + \alpha$ and S be $y = x + \beta$. We divide the cases according to $S \xrightarrow{w(S,S')} S'$, $S' \xleftarrow{w(S,S')} S$, or $S = S'$ in G. Here, we assume $S \xrightarrow{w(S,S')} S'$. The other cases are proved in a similar way. Then, D' is $y = -x + \alpha + w(D, D')$ and S' is $y = x + \beta - w(S, S')$.

We further divide the cases according to $w(D, D') < w(S, S')$, $w(D, D') = w(S, S')$, or $w(D, D') > w(S, S')$. Here, we show the case $w(D, D') < w(S, S')$. The other cases follow in a similar manner. Thus, we consider the following case, in which there may appear a number of other curves, though we omit them in the graph.



By solving the simultaneous equations $D : y = -x + \alpha$ and $S : y = x + \beta$, we find $q_1 = \frac{\alpha-\beta}{2}$ and $p_1 = \frac{\alpha+\beta}{2}$. Similarly, by solving the simultaneous equations $D' : y = -x + \alpha + w(D, D')$ and $S' : y = x + \beta - w(S, S')$, we find $q_2 = \frac{\alpha-\beta+w(D,D')+w(S,S')}{2}$ and $p_2 = \frac{\alpha+\beta+w(D,D')-w(S,S')}{2}$. This shows that the new intersection point (q_2, p_2) is well defined in the given model M.

By comparing q_1, q_2 and p_1, p_2, respectively, we have $q_2 - q_1 = \frac{w(D,D')+w(S,S')}{2} > 0$, that is, $q_1 < q_2$, and $p_2 - p_1 = \frac{w(D,D')-w(S,S')}{2} < 0$ since $w(D, D') < w(S, S')$, that is, $p_1 > p_2$. ∎

3.3 Free rides in HLGe

The **free ride** property is one of the most basic properties of diagrammatic systems that provides an account of the inferential efficacy of diagrams. By adding a certain piece of information to a diagram, the resulting diagram somehow comes to present pieces of information not contained in the given premise diagrams. Shimojima [15, 16] called this phenomenon *free ride*. By slightly extending the notion, we refer to diagrammatic objects, or the translated formulas thereof, as **free rides** if they do not appear in the given premise diagrams or sentences, but (automatically) appear in the conclusion after adding pieces of information to the given premise diagrams. The notion of free rides enables us to analyze the effectiveness of each inference rule. (Cf. [19, 21].)

As discussed in Section 2.2, our inference with graphs is conducted as follows: (1) a graph including supply and demand curves is given; (2) we shift a curve by adding a new curve; (3)

new intersection points arise; (4) we compare a new intersection point and an original one, and read off changes in price and quantity. Thus, free rides in our graphical inference are new points of intersection as well as their accompanying coordinates. We can understand this more clearly by analyzing our inference rule Apply:

$$\frac{G \quad C \xrightarrow{a} C' \quad l}{G'}\ \mathsf{Apply}$$

where C appears in G but C' does not. We compare the types, or translated formulas, of graphs of premises and the conclusion. Let $G = (\mathcal{C}, l_w, \mathcal{E}, l_p, l_q)$. Then, G' is $(\mathcal{C}', l'_w, \mathcal{E}', l'_p, l'_q)$, where:

- $\mathcal{C}' = \mathcal{C} \cup \{C \to C'\}$,
- $l'_w = l$,
- $\mathcal{E}' = \mathcal{E} \cup \{C' \cap B(q) = p \mid B \text{ is orthogonal to } C \text{ in } G\}$,
- l'_p is the linear ordering of $l_p \cup \{p \mid C' \cap B(q) = p \in \mathcal{E}'\}$,
- l'_q is the linear ordering of $l_q \cup \{q \mid C' \cap B(q) = p \in \mathcal{E}'\}$.

Observe that $\mathcal{C}'$ and l'_w are already given in the premises of Apply. In particular, the position of C' is specified by the given l $(= l'_w)$. On the other hand, the differences between $\mathcal{E}'$ and $\mathcal{E}$, l'_p and l_p, and l'_q and l_q, respectively are free rides of Apply, as they do not appear in the premises. In particular, note that the conclusion graph G' automatically obtains the linear orderings l'_p and l'_q of the y- and x-coordinates of the intersection points.

4 Abduction in economic reasoning

Our HLGe can be applied to formalize another type of reasoning, namely, abductive reasoning. Let us consider the following example similar to Example 2.1, but remove condition (3) on the shift widths of the supply and demand curves.

Example 4.1 When a new, faster computer chip is introduced, (1) demand for computers using the older, slower chips decreases (i.e., $D_2 \xleftarrow{a} D_1$). Simultaneously, (2) computer makers increase their production of computers containing the old chips in order to clear out their stocks of old chips (i.e., $S_1 \xrightarrow{b} S_2$). (3) Demand/supply decreases/increases linearly in the same rate. When the equilibrium quantity falls in response to these events, what possible explanations are there for this change?

Let $D_1 \cap S_1(q_1) = p_1$ and $D_2 \cap S_2(q_2) = p_2$. First, note that we cannot prove $q_1 > q_2$ under the given premises (1) and (2), as observed in Example 3.11. Thus, our task in this question is to find a possible explanation H such that $D_2 \xleftarrow{a} D_1$, $S_1 \xrightarrow{b} S_2, D_1 \cap S_1(q_1) = p_1, D_2 \cap S_2(q_2) = p_2, H \vdash q_1 > q_2$ holds. In Example 3.11, the two given premises (1) and (2) provide three graphs, according to whether $a < b, a = b$, or $a > b$, as shown in Fig.1. Among these three graphs, we find a graph (the third one) in which $q_1 > q_2$ holds. Thus, we know that $q_1 > q_2$ holds when $a > b$ holds for the shift widths of the demand and supply curves. Hence, we can propose $a > b$ as a possible explanation H.

This type of reasoning is called *abduction*, and frequently appears in scientific reasoning. Abduction has been extensively studied in the literature on artificial intelligence AI. In the framework of AI, abduction is usually formalized as the task of finding a hypothesis (or explanation) H that explains a given observation O under a theory (or premises) T such that O is a logical consequence of T and H, i.e., $T, H \vdash O$, and T, H are consistent. Usually, it is assumed that without H, we cannot prove O, i.e., $T \nvdash O$. Furthermore, restrictions such as "minimality" are imposed on H so that it represents "the best explanation" of the given observation O.

To solve abductive problems, the usual strategy such as resolution and proof-search to construct deductive proofs are applied. (See, for example, [13, 4] for surveys of abduction in AI.) Our strategy in this paper can be considered as a kind of model enumeration. This is because our inference in HLGe is rather model theoretic. Our inference using graphs essentially corresponds to model construction by regarding our graph as a certain kind of representative model. When there is insufficient information on the shift widths of the supply and demand curves, we enumerate all possible cases (i.e., models) by using Cases. We can then determine the required explanation from among these cases, as seen in the above example. To describe our abductive reasoning more formally, we modify Cases as follows:

$\mathsf{AbCases}$

$$\dfrac{\dfrac{l}{\overline{\overline{l_1 \vee \cdots \vee l_n}}} \qquad \dfrac{G \quad C \xrightarrow{a} C' \quad \underline{l_i}}{G_i}\,\mathsf{Apply}}{G_i}\,\mathsf{AbCases}$$

where l_i is one of the linear orderings of $l_1, \ldots, l_n$, and the underline indicates a proposed explanation.

Similarly for $C' \xleftarrow{a} C$.

In contrast to Cases, it is not necessary to consider all cases of $l_1, \ldots, l_n$, but it is sufficient to find a single case in which the given conclusion holds.

We formalize our procedure as follows. Let $\mathcal{S}, \mathcal{E}, \mathcal{O}$ be given premises, and $\mathcal{A}$ be a given conclusion or observation. Our task is to find an explanation H such that $\mathcal{S}, \mathcal{E}, \mathcal{O}, H \vdash \mathcal{A}$ holds, where we restrict H to be an ordering condition on the shift widths.

1. We construct a proof of $\mathcal{S}, \mathcal{E}, \mathcal{O} \vdash \mathcal{A}$ by using $\mathsf{AbCases}$ as well as our Apply, $\mathsf{Observe}$ and Cases for HLGe.
2. Among the applications of $\mathsf{AbCases}$, we choose a linear ordering $\underline{l_i}$ that has the maximal length, and set $H = l_i$.

Although our inference rules are not effective, especially when there are a number of possible cases to consider, we can use HLGe to formalize abductive reasoning in elementary economics in the style we employ in our actual reasoning.

5 Conclusion and future work

We have formalized heterogeneous logic with graphs in elementary economics, HLGe, by extending the usual natural deduction system. This makes it possible to apply well-developed proof-theoretic techniques such as normalization and structural analysis of proofs to the analysis of heterogeneous reasoning with graphs. We proved the soundness of HLGe, and analyzed its efficiency by applying a proof-theoretic analysis of free rides developed in [19]. Furthermore, we discussed the way in which abductive reasoning in elementary economics can also be formalized in HLGe by slightly modifying our Cases.

The completeness of the whole HLGe is not so interesting, since it already holds without graphs. An interesting problem is the characterization (through a completeness theorem) of the purely graphical fragment of HLGe, where symbolic or linguistic inferences are excluded, as we investigated for heterogeneous logic with tables in [21]. We leave such an investigation for future work.

We concentrated on a competitive market described by supply and demand models. However, extending our HLGe would enable the investigation of economic reasoning with graphs employed in various other analyses, such as a consumer's optimal consumption analysis and IS-LM analysis in macroeconomics. This is because the structures of reasoning with graphs in these instances are essentially the same as the reasoning investigated in this paper.

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